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ON THE CLOSE ANALOGY BETWEEN RADIATIVE AND CONDUCTIVE HEAT FLUX IN A FINITE SLAB

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ABSTRACT

A parallelism is established between the theoretical analyses of energy transport due to conduction and thermal radiation in a unidimensional configuration. The boundary conditions assumed correspond to two parallel walls of given temperatures separated by a finite slab of gas. A gray-gas theory of radiation is employed in which the coefficients of emission and absorption are independent of frequency; the theory of conduction is based on the Bhatnagar-Gross-Krook model of the Boltzmann equation. Under simplifying assumptions, the half-range form of the equations is modified through the use of relations drawn from continuum theory. Heat flux is predicted for all values of optical thickness or inverse Knudsen number. Comparisons with available numerical calculations based on less restrictive assumptions indicate the fluxes are given with a maximum error of a few percent yet are expressible analogously in algebraic form. The correspondence between wall accommodation and emission coefficients is exhibited. In the concluding section, generalization to other configurations yields explicit formulas for heat flux between coaxial cylinders and concentric spheres.

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INTRODUCTION

The theoretical prediction of the rate of transfer of thermal energy through a gaseous medium contained between parallel walls of different temperatures and specified physical properties is a fundamental problem in the study of heat exchange. In the theory of heat conduction in a monatomic gas one may try to determine the dependence of energy flux on the two wall temperatures, their distance apart, the mean free path length of the gas particles, and the accommodation coefficients of the walls. Similarly, in radiative transfer through an absorbing and emitting gray gas, the energy flux depends again on the wall temperatures, distance apart, the radiation (or photon) mean free path length, and on the coefficients of emission of the walls. The present note shows how, under simplifying assumptions, the flux in both cases can be predicted to a rather remarkable degree of accuracy while at the same time preserving formal similarities between the principal equations for conduction and thermal radiation.

Insofar as gray-gas radiation is concerned, the basic equations are well defined; the main contribution here is the demonstration of an effective use of the assumption of small mean free path length in the half-range form of the equations. Predictions of flux are then shown to possess reasonable merit for all optical thicknesses even though a portion of the analysis is limited to simplification inherent in the continuum or Rosseland regime.

The exact description of heat conduction for a gas with arbitrary mean free path length relies on some formulation of the Boltzmann equation. The analogy to be developed results from the use of the B-G-K model (Bhatnagar, Gross, and Krook[1]) and half-range representations of the distribution function are employed. Once the similarity between the two problems is established one is led naturally to the consideration of proper use of the continuum form of the conduction equations. The stratagem used in the radiation analysis, that is, proper employment of continuum theory, indicates the procedure and comparisons with more accurate calculations show that the predictions remain good for rarefied gases, that is, in the Knudsen regime of large mean free path length.

It is important to stress that the close analogy achieved here holds principally when attention is concentrated on flux. Temperature distributions, for example, are not the same in the two problems. Also, the distributions predicted by the approximate analysis are in both cases increasingly inaccurate as the walls are approached. It appears that flux is the grossest of the physical quantities of interest and is insensitive to approximations. More localized phenomena, such as temperature "slip" at the walls, react much more sharply to imposed approximations. Qualitatively, however, the predictions provide the proper trends and extensions of the method become apparent.

NOMENCLATURE

a	constant introduced in B-G-K model (Eq. (16))
f	distribution function
F	Maxwellian distribution (Eq. (17))
h	heat flux (Eqs. (5) and (29))
I	specific intensity of radiation (Eq. (1))
j	particle flux (Eq. (18b))

k	Boltzmann constant
K	heat conduction coefficient
L	distance between plates
m	particle mass
n	particle number density
p_{xx}	xx component of pressure tensor
T	temperature
v	particle velocity
v_x	x component of particle velocity
x	coordinate measured normal to wall
β	emission function (Eqs. (2) and (30))
ϵ	wall emission or accommodation coefficient
θ	angle relative to x axis
λ	particle or photon mean free path
μ	$\cos\theta$
ξ	$\int_0^x (dx/\lambda)$
ξ_L	$\int_0^L (dx/\lambda)$
σ	Stefan-Boltzmann constant

Subscripts

1	value in gas at $x = \xi = 0$
2	value in gas at $x = L$, $\xi = \xi_L$
M	Maxwellian value
KN	Knudsen value
w1	value at left wall
w2	value at right wall

Superscripts

+ half-range value, $0 < \mu \leqq 1$ for radiation, $0 < v_x$ for conduction

- half-range value, $-1 \leqq \mu < 0$ for radiation, $v_x < 0$ for conduction

ANALYSIS

Assume, as in Fig. 1, two walls with temperatures T_{W1} and T_{W2} at a distance apart L. In this schematic representation of the problem, conduction and radiation can be indicated simultaneously and, since algebraic similarity is to be featured, the same symbols will be used in the two cases. Thus, the parameters ϵ_1 and ϵ_2 represent either the wall accommodation coefficients of heat conduction or the wall emission coefficients for radiation. The gaseous medium between the walls has for photon or particle mean free path the local value λ. We assume also that the walls emit and reflect the incident energy diffusely.

Radiative Transport

Consider, first, the transport of thermal radiation between the walls. Distance x is measured normal to and from the left wall. Introduce now the specific intensity function which denotes the energy transmitted through a unit area (normal to the axis) in a unit time and in a unit solid angle that is inclined at an angle θ to the positive x direction. We distinguish the half-ranges $(-\pi/2 \leqq \theta \leqq \pi/2, -\pi/2 \leqq \pi - \theta \leqq \pi/2)$ of the specific intensity by the notation

$$I^+(\mu,\xi)\ , \quad 0 < \mu \leqq 1\ ; \quad I^-(\mu,\xi)\ , \quad -1 \leqq \mu < 0 \tag{1}$$

where $\mu = \cos\theta$ and $d\xi = dx/\lambda(x)$. The basic transfer equations are (see, e.g., Kourganoff[2], p. 25 et seq.)

$$\mu(dI^{\pm}/d\xi) = -I^{\pm}(\mu,\xi) + [\beta(\xi)/\pi] \tag{2}$$

The function $\beta(\xi)/\pi$ is the so-called source (or emission) function. In the absence of scattering and under the assumption that the index of refraction of the gas is unity, the condition of local thermodynamic equilibrium yields the relation $\beta(\xi) = \sigma T^4(\xi)$ where σ is the Stefan-Boltzmann constant.

Approximation in the method of solution is introduced at this point. First, one notes that the source function as well as the boundary conditions, developed later, are independent of μ when the walls emit and reflect diffusely. As a consequence, the dependence of $I^{\pm}(\mu,\xi)$ on μ is suppressed by means of an appropriate averaging. One rational way in which this averaging can be achieved involves taking first moments with respect to μ in the half-ranges. It suffices here, however, to assume some such operation has been carried out and to rewrite Eqs. (2) in the form

$$\pm\bar{\mu}(dI^{\pm}/d\xi) = -I^{\pm}(\xi) + [\beta(\xi)/\pi] \tag{3}$$

where the average value $\bar{\mu}$ is yet to be determined.

The quantities $h^{+}(\xi)$ and $h^{-}(\xi)$ are now introduced to denote the half-range energy fluxes associated with the motion of photons in the positive and negative x (or ξ) directions, respectively. Then

$$h^+(\xi) = 2\pi \int_0^1 \mu I^+ d\mu = \pi I^+(\xi) \tag{4a}$$

$$h^-(\xi) = 2\pi \int_{-1}^0 \mu I^- d\mu = -\pi I^-(\xi) \tag{4b}$$

Total flux, so defined that it is positive when the net flow of energy in the x direction is positive, is $h(\xi)$ where

$$h(\xi) = h^+(\xi) + h^-(\xi) \tag{5}$$

From Eqs. (3) and (4) one derives the relations

$$\bar{\mu}(dh^+/d\xi) = \beta(\xi) - h^+(\xi) \tag{6a}$$

$$\bar{\mu}(dh^-/d\xi) = \beta(\xi) + h^-(\xi) \tag{6b}$$

Addition and subtraction of Eqs. (6) together with the definition introduced in Eq. (5) yields

$$\bar{\mu}(dh/d\xi) = 2\beta(\xi) - [h^+(\xi) - h^-(\xi)] \tag{7a}$$

$$\bar{\mu}[d(h^+ - h^-)/d\xi] = -h(\xi) \tag{7b}$$

For the problem being considered, conservation of energy requires that flux be a constant. From Eq. (7a), therefore,

$$\beta(\xi) = [h^+(\xi) - h^-(\xi)]/2 \tag{8}$$

and the fundamental equations are

$$dh(\xi)/d\xi = 0 \tag{9a}$$

$$d\beta(\xi)/d\xi = -h(\xi)/2\bar{\mu} \tag{9b}$$

Equation (9a) is independent of the approximation we have employed and the arbitrariness in the choice of $\bar{\mu}$ appears only in the second relation. The decision as to what value to assign to $\bar{\mu}$ is now resolved by insisting that Eq. (9b) must agree with the expression for flux in the limiting regime of an optically thick medium. This final assumption fixes the value $2\bar{\mu} = 4/3$ since independent study (e.g., Kourganoff[2]) of radiative transfer for $\lambda \ll L$ leads to the condition $d\beta/d\xi = -3h/4$ at any point ξ sufficiently distant from an imposed wall or boundary condition. Since the approximate analysis is based on the use of isotropic specific intensity, that is, no dependence of $I^{\pm}(\mu,\xi)$ on μ, we retain a certain logical consistency by adjusting the arbitrary constant to conform with a regime in which the isotropic condition does apply. Very accurate numerical solutions are available for special values of the physical parameters, and an *a posteriori* check of the results for these cases will be given later.

The boundary conditions express the equality of outwardly directed flux at each wall to the sum of the wall emission and the reflected portion of the inwardly directed flux at the wall. Since the reflectivity coefficient of an opaque wall is 1 minus the absorption (or emission) coefficient, one has

$$h_1^+ = \epsilon_1\beta_{W1} - (1 - \epsilon_1)h_1^- \qquad (10a)$$

$$-h_2^- = \epsilon_2\beta_{W2} + (1 - \epsilon_2)h_2^+ \qquad (10b)$$

Here, subscripts 1 and 2 refer to conditions in the medium at $x = 0$ and $x = L$ and subscripts w1 and w2 refer to wall conditions at the same positions. Combination of these relations leads to

$$h\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) = \beta_{w1} - \beta_{w2} + \frac{h_2^+ - h_2^-}{2} - \frac{h_1^+ - h_1^-}{2} \tag{11}$$

Equation (11) retains half-range fluxes but from Eq. (8) it can be written alternatively as

$$h\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) = \beta_{w1} - \beta_{w2} + \beta_2 - \beta_1 \tag{12}$$

The integration of Eqs. (9) subject to the boundary conditions is now a straightforward process. The constant value of flux, h, is independent of ξ and a function only of the parameters T_{w1}, T_{w2}, ϵ_1, ϵ_2, and the optical thickness ξ_L where $\xi_L = \int_0^L \lambda^{-1}(x)dx$. When $\epsilon_1 = \epsilon_2 = 1$, $\xi_L = 0$, flux is $\sigma T_{w1}^4 - \sigma T_{w2}^4$ and is the heat flux associated with black wall conditions. The end results may be written in the form

$$\beta(\xi) = h\left[b - \frac{3}{4}\left(\xi - \frac{1}{2}\xi_L\right)\right] \tag{13}$$

where

$$b = \frac{\sigma T_{w1}^4\left(\frac{1}{\epsilon_2} - \frac{1}{2} + \frac{3}{8}\xi_L\right) + \sigma T_{w2}^4\left(\frac{1}{\epsilon_1} - \frac{1}{2} + \frac{3}{8}\xi_L\right)}{\sigma T_{w1}^4 - \sigma T_{w2}^4} \tag{14}$$

and

$$h = \frac{\sigma T_{w1}^4 - \sigma T_{w2}^4}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) + \frac{3}{4}\xi_L} \tag{15}$$

Equation (13) gives a linear expression for the emission function and represents a rough approximation to the actual distribution of $\sigma T^4(\xi)$. Equation (15) is the principal objective of the analysis. Its accuracy, when compared with available and more exact numerical calculations, is surprisingly good. Figure 2 shows $h/(\sigma T_{w1}^{\ 4} - \sigma T_{w2}^{\ 4})$ as a function of ξ_L and ϵ for the case of physically similar walls ($\epsilon_1 = \epsilon_2$). Solid lines correspond to the predictions of Eq. (15). The dashed lines were taken from the work of Heaslet and Fuller[3] where an iterative method was used to solve the basic integral equation of radiation theory for the same boundary conditions. Probstein[4] has previously noted that for black walls ($\epsilon_1 = \epsilon_2 = 1$) the agreement with an exact numerical analysis carried out by Usiskin and Sparrow[5] is quite satisfactory. The topmost curve is Probstein's result.

Conductive Transport

Heat conduction between the walls and through a monatomic gas remains to be considered. If Eqs. (8), (9), and the boundary conditions (10) are accepted as fundamental to the prediction of heat flux, a parallel development is possible. Preliminary to this, the specific forms of the kinetic equations need to be introduced.

The B-G-K[1] idealization of the Boltzmann equation for the present problem can be expressed as

$$v_x(df/dx) = (a\bar{v}/\lambda)(F - f) \tag{16}$$

where

$$F = n(m/2\pi kT)^{3/2} \exp(-mv^2/2kT) \tag{17}$$

(For a valuable critique of this idealization see Liepmann, Narasimha, and Chahine.[6]) The quantity v_x is the x component of particle velocity v, and $\bar{v}$ is the average particle velocity for the Maxwellian distribution, F. The quantity λ is related to the particle mean free path length, also based on F, and a is a proportionality constant to be determined by requiring the solution for heat flux to be exact at the continuum limit. Thus λ and a may be functions of n and T for arbitrary intermolecular force laws.

The values of n,T and other quantities of interest are given in terms of moments of the distribution function f by the relations

$$n = \int f \, d\vec{v} \quad \text{(number density)} \tag{18a}$$

$$j = \int v_x f \, d\vec{v} \quad \text{(flux of particles)} \tag{18b}$$

$$p_{xx} = \int m v_x^2 f \, d\vec{v} \quad \text{(xx component of pressure tensor)} \tag{18c}$$

$$(3/2)nkT = \int (1/2) m v^2 f \, d\vec{v} \quad \text{(internal energy density)} \tag{18d}$$

$$h = \int (1/2) m v_x v^2 f \, d\vec{v} \quad \text{(total energy flux)} \tag{18e}$$

Upon multiplication of Eq. (16) by the appropriate collisional invariant (1, mv_x, or mv^2) and integration over particle velocities, we get the conservation equations $\partial j/\partial x = 0$, $\partial p_{xx}/\partial x = 0$, $\partial h/\partial x = 0$, Noting that j is zero at the walls, these can be written

$$j = 0 \tag{19a}$$

$$p_{xx} = \text{const} = \text{force per unit area on walls} \tag{19b}$$

$$h = \text{const} \tag{19c}$$

The kinetic theory counterpart of the foregoing simplified treatment of radiative transport entails the definition of two half-range distribution functions $f^{\pm}$ for particles with positive or negative values of v_x, and the two distributions are taken to be Maxwellian, that is,

$$f^{\pm} = 2n^{\pm}(m/2\pi kT^{\pm})^{3/2} \exp(-mv^2/2kT^{\pm}) \tag{20}$$

Substitution of this into Eqs. (18a) to (18d) yields

$$n = n^{+} + n^{-} \tag{21a}$$

$$j = n^{+}(2kT^{+}/\pi m)^{1/2} - n^{-}(2kT^{-}/\pi m)^{1/2} = 0 \tag{21b}$$

$$p_{xx} = n^{+}kT^{+} + n^{-}kT^{-} = \text{const} \tag{21c}$$

$$nkT = n^{+}kT^{+} + n^{-}kT^{-} = p_{xx} \tag{21d}$$

The equalities on the right follow from the use of the conservation relations (19a), (19b) and from combination of (21c) with (21d).

The following simplified treatment of the kinetic theory heat conduction problem can be cast in a form that is independent of the intermolecular force law. To that end, we choose a definition of λ based on the heat conduction coefficient and the hard sphere mean free path, $\lambda_{HS} = (\sqrt{2}\ \pi d^2 n)^{-1}$ where d is particle diameter. For hard spheres, the heat conduction coefficient is

$$K_{HS} = 1.02513\ (75/64d^2)(k^3T/\pi m)^{1/2}$$

see Chapman and Cowling,[7] chapter 10. Alternatively, an approximate evaluation of the numerical coefficient taken from Jeans[8] leads to

$$K_{HS} = (15/4\pi d^2)(k^3T/\pi m)^{1/2}$$

The difference between the two expressions is less than 1 percent, and we shall adopt the latter because of its more compact form. In a later development the parameter a appearing in Eq. (16) will be evaluated by consideration of the form of the equations in the continuum limit corresponding to vanishingly small mean free path. It can be seen that this procedure is independent of the form of the intermolecular force law if an effective mean free path is defined as

$$\lambda = (K/K_{HS})\lambda_{HS} = K/(15/4)kn(2kT/\pi m)^{1/2} \tag{22}$$

where K is the continuum heat-conduction coefficient. If momentum transport were considered, a separate effective mean free path would have to be defined for that case, but in the present conduction problem this is not necessary.

It is of interest to compare our effective mean free path for conduction, Eq. (22), with values that have been used in the literature. For Maxwellian molecules, the heat-conduction coefficient is[7]

$$K_M = [5/4\pi A_2(5)](2/m\kappa)^{1/2}\, k^2T$$

where $A_2(5) = 0.436$ and κ is the constant in the inverse fifth power force law. In the study of heat flow by Ziering[10] the solution for Maxwellian molecules is expressed in terms of a mean free path λ_M defined by Maxwell as

$$\lambda_M = 1/[2\pi A_2(5)(\kappa/kT)^{1/2}\, n]$$

The heat-conduction coefficient can be expressed in terms of this parameter as

$$K_M = (5kn/2)(2kT/m)^{1/2}\,\lambda_M$$

By substitution in Eq. (22), we find

$$\lambda = (2\sqrt{\pi}/3)\lambda_M \qquad \text{(for Maxwellian molecules)}$$

When expressed in terms of λ, our results are independent of the form of the intermolecular force law. However, for comparison with experiment or other results in the literature this factor must be considered. For example, Lavin and Haviland[11] define Knudsen numbers for hard spheres and Maxwellian molecules in terms of an average density

$$\bar{\rho} = (m/L)\int_0^L n\,dr \tag{23}$$

It is apparent that $\bar{\rho}$ is a measure of the amount of gas between the walls, which can be inferred from experimentally measurable quantities. The solution to be given later is expressed in terms of the variable ξ, which is related to x and λ by

$$\xi = \int_0^x dx_1/\lambda(x_1) \tag{24a}$$

or

$$d\xi = dx/\lambda \tag{24b}$$

Rearrangement and substitution of Eqs. (22) and (24b) into Eq. (23) yields

$$\bar{\rho} = \frac{4}{15}\frac{m}{Lk}\int_0^{\xi_L}\frac{K\,d\xi}{(2kT/\pi m)^{1/2}} \tag{25}$$

The solution to be given later leads to a temperature distribution $T(\xi)$. When the heat-conduction coefficient K is specified as a function of T, the average density can be evaluated according to Eq. (25).

Through use of Eq. (24b), Eq. (16) becomes

$$v_x(df/d\xi) = a\bar{v}(F - f) \tag{26}$$

which we choose to rewrite in terms of half-range functions ($v_x > 0$, $v_x < 0$) as

$$(v_x/a\bar{v})(df^{\pm}/d\xi) = -f^{\pm}(v,\xi) + F(v,\xi) \tag{27}$$

The half-range energy fluxes are, respectively,

$$h^{+}(\xi) = 2n^{+}kT^{+}(2kT^{+}/\pi m)^{1/2} \tag{28a}$$

$$h^{-}(\xi) = -2n^{-}kT^{-}(2kT^{-}/\pi m)^{1/2} \tag{28b}$$

and total flux $h(\xi)$ is

$$h(\xi) = h^{+}(\xi) + h^{-}(\xi) \tag{29}$$

Conformity between the present analysis and the approximate theory used previously is preserved if Eq. (27) is rewritten as flux equations with an appropriate averaging in velocity space. Thus, if Eq. (27) is multiplied by $mv_xv^2/2$ and integrations carried out over the half-range particle velocities,

$$(\overline{v_x/a\bar{v}})(dh^+/d\xi) = \beta(\xi) - h^+(\xi) \tag{30a}$$

$$(\overline{v_x/a\bar{v}})(dh^-/d\xi) = \beta(\xi) + h^-(\xi) \tag{30b}$$

where $\beta(\xi) = nkT(2kT/\pi m)^{1/2}$ and the coefficients in the left members of the differential equations are yet to be fixed. The direct analogy with Eqs. (6) is obvious and, as in Eqs. (7), combination of the flux equations yields

$$(\overline{v_x/a\bar{v}})(dh/d\xi) = 2\beta(\xi) - [h^+(\xi) - h^-(\xi)] \tag{31a}$$

$$(\overline{v_x/a\bar{v}})[d(h^+ - h^-)/d\xi] = -h(\xi) \tag{31b}$$

As noted in Eq. (19c), total flux is constant, so

$$\beta(\xi) = [h^+(\xi) - h^-(\xi)]/2 \tag{32}$$

and the fundamental equations become

$$dh(\xi)/d\xi = 0 \tag{33}$$

$$d\beta(\xi)/d\xi = -h(\xi)/2(\overline{v_x/a\bar{v}}) \tag{34}$$

Precision in the statement of the latter relation now appears if, as previously, one expresses it in conformity with the dictates of continuum theory and small mean free path length. With the aid of Eqs. (21c) and (21d) β can be expressed as

$$\beta(\xi) = nkT(2kT/\pi m)^{1/2} = p_{xx}(2kT/\pi m)^{1/2} \qquad (35)$$

where p_{xx} is constant. Differentiation and rearrangement leads to

$$d\beta/d\xi = (nk/2)(2kT/\pi m)^{1/2}\,(dT/d\xi)$$

The quantity $dT/d\xi$ can be eliminated in favor of h by use of the continuum relation $h = -K(dT/dx)$. By substitution of Eq. (22) to remove K, and use of Eq. (24b), we get

$$h = -(15/4)nk(2kT/\pi m)^{1/2}\,(dT/d\xi)$$

Combination of this with the expression for $d\beta/d\xi$ above yields

$$d\beta(\xi)/d\xi = -2h/15 \qquad (36)$$

where $\beta(\xi)$ is given explicitly in Eq. (35). In the approximate analysis, $T^4(\xi)$ is a linear function for radiative transport and $T^{1/2}(\xi)$ is a linear function for conductive transport.

It remains to introduce the boundary conditions. More detailed consideration is required than was used earlier in Eqs. (10). We write these conditions as

$$f_1^+(v_x,v) = \epsilon_1 F_{w1} + (1 - \epsilon_1) f_1^-(-v_x,v) , \qquad v_x > 0 \tag{37a}$$

$$f_2^-(v_x,v) = \epsilon_2 F_{w2} + (1 - \epsilon_2) f_2^+(-v_x,v) , \qquad v_x < 0 \tag{37b}$$

Here, $f_1^{\pm}$ and $f_2^{\pm}$ are the half-range distribution functions in the gas evaluated, respectively, at the wall positions. The distribution functions associated with diffuse emission at the walls are

$$F_{w1} = A_1(m/2\pi kT_{w1})^{3/2} \exp(-mv^2/2kT_{w1}) \tag{38a}$$

$$F_{w2} = A_2(m/2\pi kT_{w2})^{3/2} \exp(-mv^2/2kT_{w2}) \tag{38b}$$

where T_{w1} and T_{w2} are the two wall temperatures. The values of A_1 and A_2 are yet to be determined; the parameters ϵ_1 and ϵ_2 are the wall accommodation coefficients.

The counterparts of Eqs. (10) are

$$h_1^+ = \epsilon_1 kA_1 T_{w1}(2kT_{w1}/\pi m)^{1/2} - (1 - \epsilon_1) h_1^- \tag{39a}$$

$$-h_2^- = \epsilon_2 kA_2 T_{w2}(2kT_{w2}/\pi m)^{1/2} + (1 - \epsilon_2) h_2^+ \tag{39b}$$

Through the use of Eq. (32), one has

$$h\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) = \left(\frac{2k^3}{\pi m}\right)^{1/2}\left(A_1 T_{w1}{}^{3/2} - A_2 T_{w2}{}^{3/2}\right) + \beta_2 - \beta_1 \tag{40}$$

The solution of the differential Eq. (36) is expressible as

$$\beta(\xi) = h[b - (2/15)(\xi - \xi_L/2)] \tag{41}$$

From the boundary conditions one gets

$$b = \frac{\left[A_1 T_{w1}^{3/2}\left(\frac{1}{\epsilon_2} - \frac{1}{2} + \frac{\xi_L}{15}\right) + A_2 T_{w2}^{3/2}\left(\frac{1}{\epsilon_1} - \frac{1}{2} + \frac{\xi_L}{15}\right)\right]}{\left(A_1 T_{w1}^{3/2} - A_2 T_{w2}^{3/2}\right)} \tag{42}$$

$$h = \frac{\left(\frac{2k^3}{\pi m}\right)^{1/2}\left(A_1 T_{w1}^{3/2} - A_2 T_{w2}^{3/2}\right)}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) + \frac{2\xi_L}{15}} \tag{43}$$

The final relation gives flux in terms of the wall emission characteristics but a difference exists here between conduction and radiation. In the latter case the emission from the walls is purely a function of wall temperature. One distinguishing difference is associated with the fact that photons need not be conserved in absorption and emission at the walls. Thus, a more detailed balancing of conditions is necessary for the gas particles, and the parameters A_1 and A_2 are constrained by known relations. The additional calculations are most easily carried out if a perturbation analysis is used in which the percentage change in temperature is not excessive. To this end we introduce $\Delta T = (T_{w1} - T_{w2})/2$ and assume $2\Delta T/[(1/2)(T_{w1} + T_{w2})] \leq 1$.

From Eqs. (21a) and (21b)

$$\left.\begin{aligned} n^+(\xi) &= n(\xi)[T^-(\xi)]^{1/2}/\left\{[T^+(\xi)]^{1/2} + [T^-(\xi)]^{1/2}\right\} \\ n^-(\xi) &= n(\xi)[T^+(\xi)]^{1/2}/\left\{[T^+(\xi)]^{1/2} + [T^-(\xi)]^{1/2}\right\} \end{aligned}\right\} \tag{44}$$

and from Eq. (21d)

$$T(\xi) = [T^+(\xi)T^-(\xi)]^{1/2}$$

Temperature is therefore equal to the geometric mean of the half-range temperatures. If the deviations in the temperatures are small enough that the geometric and arithmetic means can, to a first order, be equated, one has $2T(\xi) \approx [T^+(\xi)T^-(\xi)]^{1/2} + [T^+(\xi) + T^-(\xi)]/2$ and

$$2[T(\xi)]^{1/2} \approx [T^+(\xi)]^{1/2} + [T^-(\xi)]^{1/2} \tag{45}$$

Multiplication by v_x and integration of Eqs. (37a) and (37b) over the half-range velocity spaces leads to

$$\begin{aligned} A_1(T_{w1})^{1/2} &= 2n^+(0)[T^+(0)]^{1/2} \\ &= 2n(0)T(0)/\{[T^+(0)]^{1/2} + [T^-(0)]^{1/2}\} \approx n(0)[T(0)]^{1/2} \end{aligned} \tag{46a}$$

$$\begin{aligned} A_2(T_{w2})^{1/2} &= 2n^+(\xi_L)[T^+(\xi_L)]^{1/2} \\ &= 2n(\xi_L)T(\xi_L)/\{[T^+(\xi_L)]^{1/2} + [T^-(\xi_L)]^{1/2}\} \approx n(\xi_L)[T(\xi_L)]^{1/2} \end{aligned} \tag{46b}$$

Equation (43) is then rewritten as

$$h = \frac{\left(\dfrac{2k^3}{\pi m}\right)^{1/2} \{n(0)[T(0)]^{1/2}\, T_{w1} - n(\xi_L)[T(\xi_L)]^{1/2}\, T_{w2}\}}{\left(\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_2} - 1\right) + \dfrac{2\xi_L}{15}} \tag{47}$$

Further reduction is possible through use of the relations

$$p_{xx} = kn(0)T(0) = kn(\xi_L)T(\xi_L) = \text{const}$$

$$p_{xx}[2kT(\xi)/\pi m]^{1/2} = p_{xx}[2kT(\xi_L/2)/\pi m]^{1/2} - (2h/15)(\xi - \xi_L/2)$$

the latter expression being merely a re-expression of Eqs. (41) and (35). Equation (47) then becomes

$$h = \frac{p_{xx}\left(\frac{2k}{\pi m}\right)^{1/2}}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) + \frac{2\xi_L}{15}} \left\{ \frac{T_{w1}}{\left[T\left(\frac{\xi_L}{2}\right)\right]^{1/2} + \frac{h\xi_L}{15p_{xx}} \left(\frac{\pi m}{2k}\right)^{1/2}} \right.$$

$$\left. - \frac{T_{w2}}{\left[T\left(\frac{\xi_L}{2}\right)\right]^{1/2} - \frac{h\xi_L}{15p_{xx}} \left(\frac{\pi m}{2k}\right)^{1/2}} \right\} \qquad (48)$$

It is convenient at this point to introduce the Knudsen flux, h_{KN}, which applies at $\epsilon_1 = \epsilon_2 = 1$, $\xi_L = 0$. Under these conditions $T^+(\xi) = T_{w1}$, $T^-(\xi) = T_{w2}$, and direct calculation yields

$$h_{KN} = \frac{4p_{xx}}{(\sqrt{T_{w1}} + \sqrt{T_{w2}})} \left(\frac{2k}{\pi m}\right)^{1/2} \Delta T \qquad (49a)$$

or, using Eq. (45),

$$h_{KN} = 2p_{xx} \left[\frac{2k}{\pi m(T_{w1}T_{w2})^{1/2}}\right]^{1/2} \Delta T \qquad (49b)$$

To the order of accuracy of the present analysis these results are indistinguishable but we shall use the first in view of its more straightforward derivation.

Algebraic manipulation of Eq. (48) now leads to the result

$$\frac{h\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1 + \frac{2\xi_L}{15}\right)}{h_{KN}\left[\frac{(T_{w1})^{1/2} + (T_{w2})^{1/2}}{2\sqrt{T(\xi_L/2)}}\right]}$$

$$= \frac{1 - \frac{2h\xi_L}{15h_{KN}}\left[\frac{2\sqrt{T(\xi_L/2)}}{(T_{w1})^{1/2} + (T_{w2})^{1/2}}\right]\left[\frac{T_{w1} + T_{w2}}{2T(\xi_L/2)}\right]}{1 - \left(\frac{2h\xi_L}{15h_{KN}}\right)^2\left[\frac{(T_{w1})^{1/2} + (T_{w2})^{1/2}}{2\sqrt{T(\xi_L/2)}}\right]^2\left\{\frac{2(\Delta T)^{1/2}}{(T_{w1})^{1/2} + (T_{w2})^{1/2}}\right\}^4} \tag{50}$$

Two formulas of interest follow from this relation. First, we note that since $\sqrt{T(\xi)}$ is a linear function the bracketed terms are, respectively,

$$\frac{(T_{w1})^{1/2} + (T_{w2})^{1/2}}{\sqrt{T(0)} + \sqrt{T(\xi_L)}}, \qquad \frac{2(T_{w1} + T_{w2})}{[\sqrt{T(0)} + \sqrt{T(\xi_L)}]^2}$$

When $\epsilon_1 = \epsilon_2 = \epsilon$ it may be conjectured that the ratios of the averaged wall temperatures and the averaged gas temperatures at the wall are nearly 1. In this case, one gets

$$\frac{h}{h_{KN}}\left(\frac{2}{\epsilon} - 1 + \frac{2\xi_L}{15}\right) = \frac{1 - 2\frac{\xi_L h}{15h_{KN}}}{1 - \left(\frac{2\xi_L h}{15h_{KN}}\right)^2\left\{\frac{2(\Delta T)^{1/2}}{(T_{w1})^{1/2} + (T_{w2})^{1/2}}\right\}^4} \tag{51}$$

Second, more rigorously, we can restrict the analysis to a first-order theory involving the small parameter $2\Delta T/(T_{w1} + T_{w2})$. Equation (50) then reduces to

$$h = \frac{h_{KN}}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) + \frac{4\xi_L}{15}} \tag{52}$$

This latter expression is precisely analogous to radiation flux given in Eq. (15). The role of h_{KN}, which is h evaluated at $\epsilon_1 = \epsilon_2 = 1$, $\xi_L = 0$ is the same as that of black body flux $\sigma T_{w1}^4 - \sigma T_{w2}^4$; the difference in the coefficients of ξ_L is attributable to the coefficients appearing in the continuum forms of the transport equations. Figure 3 shows a comparison between this result and detailed calculations of Gross and Ziering[9] for $\epsilon_1 = \epsilon_2 = 1$. The degree of excellence is not as good as that shown in Fig. 2 and, in view of the added simplifications used to achieve this expression, the difference is not surprising. In particular, the starting slope of the approximate curve fails to conform with the more nearly exact calculations. The usefulness of the result in predicting the magnitude of flux is, however, apparent.

Figure 4 shows a comparison between predictions of slip temperature T_s. By definition, when $\epsilon_1 = \epsilon_2 = \epsilon$,

$$T_s/\Delta T = 1 + [T(\xi_L) - T(0)]/2\,\Delta T \tag{53}$$

and in the present approximate theory one has

$$\frac{T_s}{\Delta T} = \frac{\left(\frac{2}{\epsilon} - 1\right)}{\left(\frac{2}{\epsilon} - 1\right) + \frac{4\xi_L}{15}} \tag{54}$$

The comparisons were made to correspond to the results of Gross and Ziering,[9] and necessarily were applied at $\epsilon = 1$.

CONCLUDING REMARKS

The preceding analysis has shown that heat flux may be calculated approximately through a simplified analysis in which differential equations from continuum theory are solved subject to boundary conditions involving temperature discontinuities at the walls. It is also possible to characterize the final results in another way. To this end we limit attention to fixed values of mean free path λ. Then one notes that the predicted heat flux has the form

$$\frac{h}{h_{\epsilon_1=\epsilon_2=1,\lambda=\infty}} = \frac{1}{A + \frac{B}{\lambda}} \tag{53}$$

where A and B are associated, respectively, with the extreme conditions $\lambda \gg 1$ and $\lambda \ll 1$. If this restatement of the end result is given a definitive status, generalizations to other cases become possible so long as A is provided by a free-molecule or optically-thin analysis and B can be calculated from continuum theory. The asymptotic behavior of Eq. (53) is correct for $1/\lambda$ large and the magnitude of h is exact at $1/\lambda \approx 0$. The one-dimensional analysis and Figs. 2 and 3 indicate that the region of least accuracy will appear at moderately small values of mean free path length where distance is, of course, measured in terms of a characteristic geometric length.

Two additional results follow immediately. Consider, first, two coaxial cylinders with radii R_1, R_2 $(R_1 < R_2)$, temperatures T_{W1}, T_{W2}, and accommodation coefficients or emissivities ϵ_1, ϵ_2. If h_1 is heat flux at the inner cylinder one has

RADIATION
$$h_1 = \frac{\sigma\left(T_{w1}^{\ 4} - T_{w2}^{\ 4}\right)}{\frac{1}{\epsilon_1} + \frac{R_1}{R_2}\left(\frac{1}{\epsilon_2} - 1\right) + \frac{3}{4}\frac{R_1}{\lambda}\,\ln\frac{R_2}{R_1}} \qquad (54a)$$

CONDUCTION
$$h_1 = \frac{h_{KN}}{\frac{1}{\epsilon_1} + \frac{R_1}{R_2}\left(\frac{1}{\epsilon_2} - 1\right) + \frac{4}{15}\frac{R_1}{\lambda}\,\ln\frac{R_2}{R_1}} \qquad (54b)$$

In Eq. (54a) the value at $\lambda = \infty$ is provided by Christiansen's formula as discussed, for example, by Jakob,[12] page 5, or by Jensen.[13] A derivation of the corresponding term in Eq. (54b) can be found, for example, in Kennard.[14] The logarithmic terms arise from the fact that the continuum equations in cylindrical geometry require flux h to vary inversely with radius.

Consider, next, two concentric spheres with radii R_1, R_2 and, again, let h_1 be heat flux at the inner surface. Equation (53) then yields

RADIATION
$$h_1 = \frac{\sigma\left(T_{w1}^{\ 4} - T_{w2}^{\ 4}\right)}{\frac{1}{\epsilon_1} + \frac{R_1^2}{R_2^2}\left(\frac{1}{\epsilon_2} - 1\right) + \frac{3}{4}\frac{R_1}{R_2}\frac{R_2 - R_1}{\lambda}} \qquad (55a)$$

CONDUCTION
$$h_1 = \frac{h_{KN}}{\frac{1}{\epsilon_1} + \frac{R_1^2}{R_2^2}\left(\frac{1}{\epsilon_2} - 1\right) + \frac{4}{15}\frac{R_1}{R_2}\frac{R_2 - R_1}{\lambda}} \qquad (55b)$$

Christiansen's formula again provides the appropriate relation in Eq. (55a). For spherical geometry the continuum theory requires h to vary inversely as radius squared.

It is to be noted that when $R_2 - R_1 = L$ and R_2 increases indefinitely the above results reduce to the formulas derived previously for the case of parallel walls. When $\epsilon_1 = \epsilon_2 = 1$, Eq. (54b) reduces to a form derived by Lees and Liu.[15] A further specialization of interest also follows from Eqs. (55), namely, the case of a single sphere conducting or radiating to an infinite gaseous medium. Here the external radius R_2 becomes arbitrarily large while R_1 is held fixed. A finite, nonzero value of flux is predicted. In the idealized case of the cylindrical geometry or in the purely one-dimensional configuration the theory predicts zero flux for finite values of T_{W1} and T_{W2} when the distance between the walls becomes infinite. This anomalous behavior stems from the change in the steady-state continuum equations as the dimensionality of the configuration changes. The differences between the single sphere and the other cases are closely analogous mathematically to contrasting results that arise in attempting to predict the drag of the same geometric shapes in a Stokes flow analysis. The latter analysis, however, is based on additional idealizations of the basic physical equations.

In a direct derivation of Eqs. (54) and (55) by the methods of this paper, the recasting of the differential operators in the governing equations offers no difficulty. The boundary conditions require more detailed investigation, however, and the initial study of the finite slab shows that the kinetic theory analysis is the more difficult to carry through and should be limited to small temperature differences. The approach adopted here is essentially one of demanding that the boundary conditions must be exact for limiting values of λ. Equation (53) is thus an interpolation formula with known end conditions. Further improvement would involve increased accuracy in the prediction of the gradient $dh/d\lambda^{-1}$ at $\lambda^{-1} = 0$.

The attainment of an acceptable analogy without great loss in accuracy of predictions has been the theme of this paper and was the motivation for the investigation. It is important, also, to stress that the first-order predictions of temperature distribution are of considerable value as starting points for more exact iterative calculations. It is not possible to maintain a continuing analogy, since differences appear in the character of the influence functions in the integral-equation formulations,[6] for example, but this does not detract from the utility of the first-order results. Two recent investigations may be mentioned in which predictions from a continuum-oriented theory are used to calculate more exact results. The first of these is the work of Heaslet and Fuller;[3] radiative transport and temperature distributions between parallel plates are determined iteratively, starting with the approximation of the emission function given in Eq. (13). In the work of Liepmann, Narasimha, and Chahine[6] the structure of a plane shock layer is studied by means of the B-G-K model. Success in carrying out an iterative calculation was attributed in part to the use of the Navier-Stokes equations to get a starting solution. This approach is consistent with the type of approximation we have advocated here.

The methods used in the derivation of the present approximate solutions can be generalized in other directions. For example, it may be possible, by slight modifications, to include complicating factors such as simultaneous transport by radiation and conduction in gas mixtures including polyatomic molecules.

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FIGURE LEGENDS

Fig. 1.- Sketch showing parallel walls separated by gaseous medium.

Fig. 2.- Dimensionless radiative flux between similar walls as a function of wall emissivity and optical thickness.

Fig. 3.- Dimensionless conductive flux as a function of inverse Knudsen number, $\epsilon_1 = \epsilon_2 = 1$.

Fig. 4.- Dimensionless temperature slip as a function of inverse Knudsen number, $\epsilon_1 = \epsilon_2 = 1$.

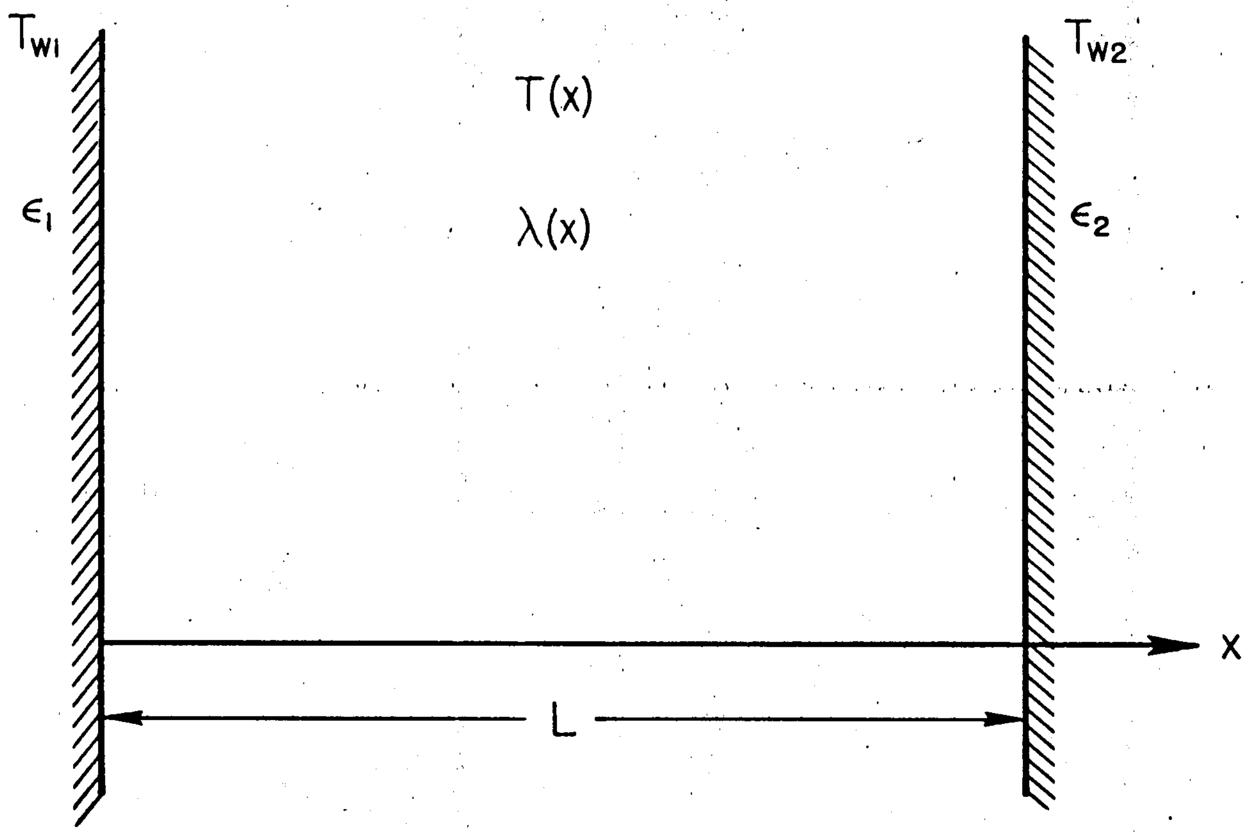

Fig. 1

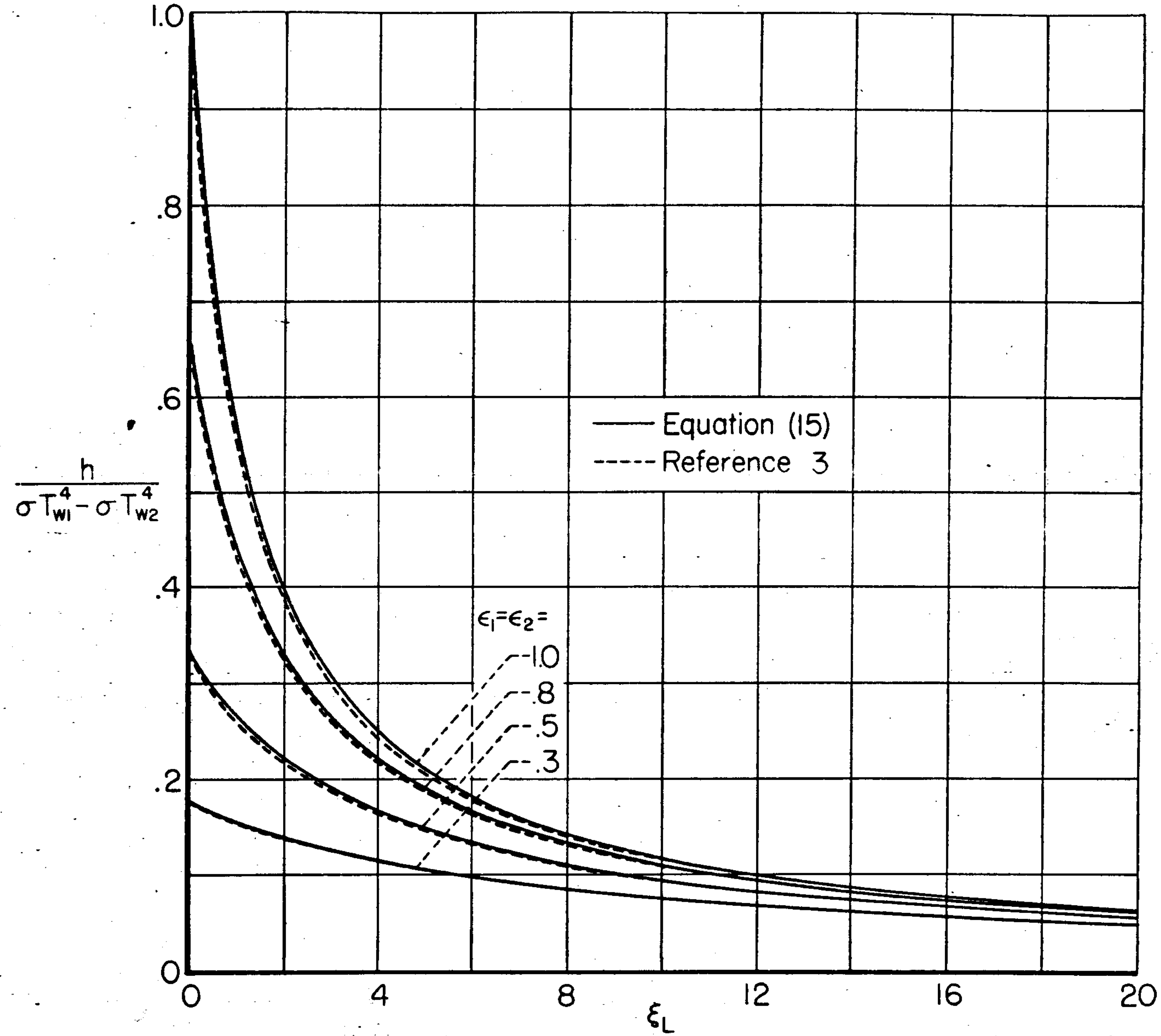
1.0
.8
.6
.4
.2
0
$\frac{h}{\sigma T_{w1}^4 - \sigma T_{w2}^4}$
Equation (15)
Reference 3
$\epsilon_1 = \epsilon_2 =$
1.0
.8
.5
.3
0
4
8
12
16
20
ξ_L

Fig. 2

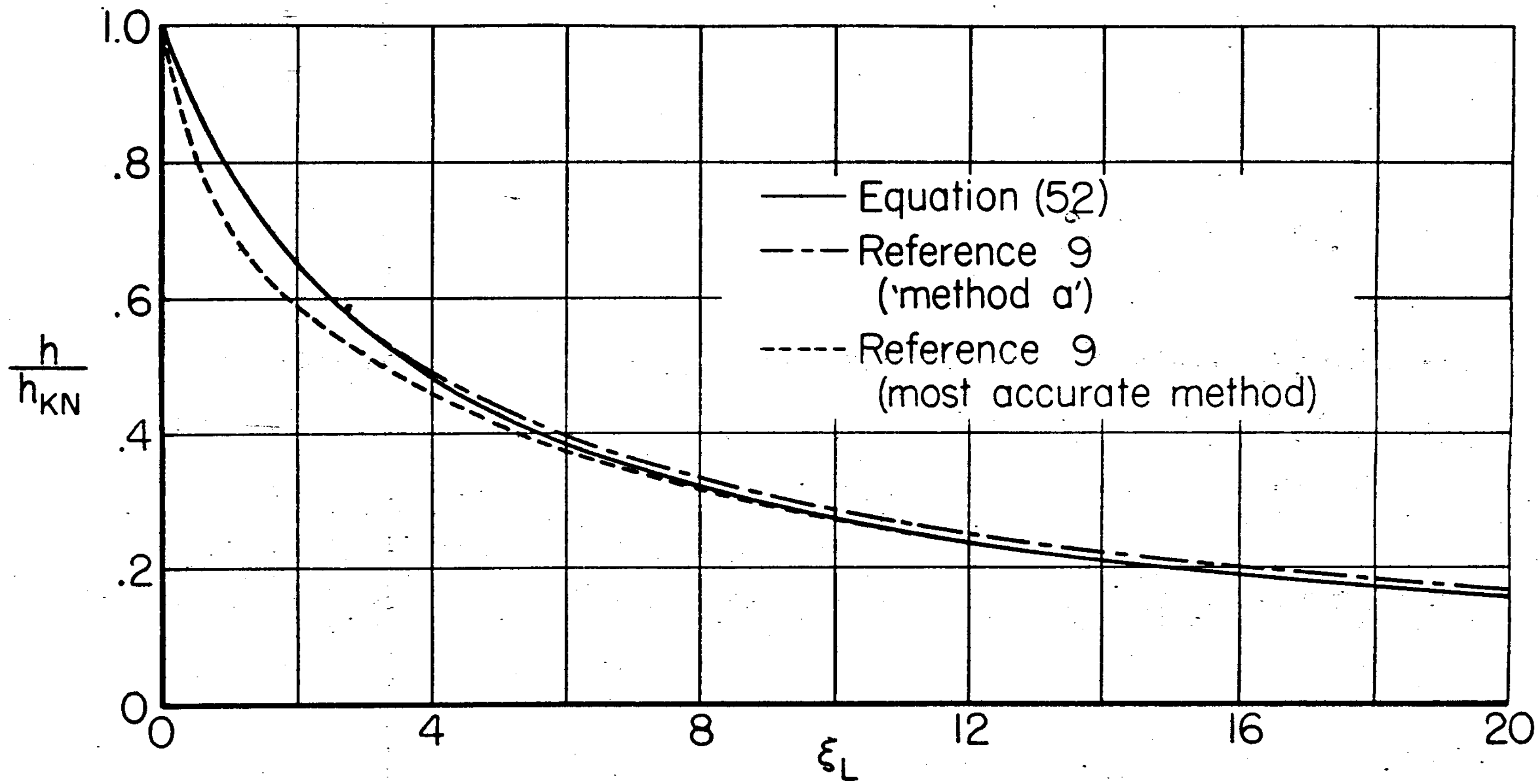
1.0
.8
.6
.4
.2
0
h/hKN
0
4
8
12
16
20
ξL
Equation (52)
Reference 9
('method a')
Reference 9
(most accurate method)

Fig. 3

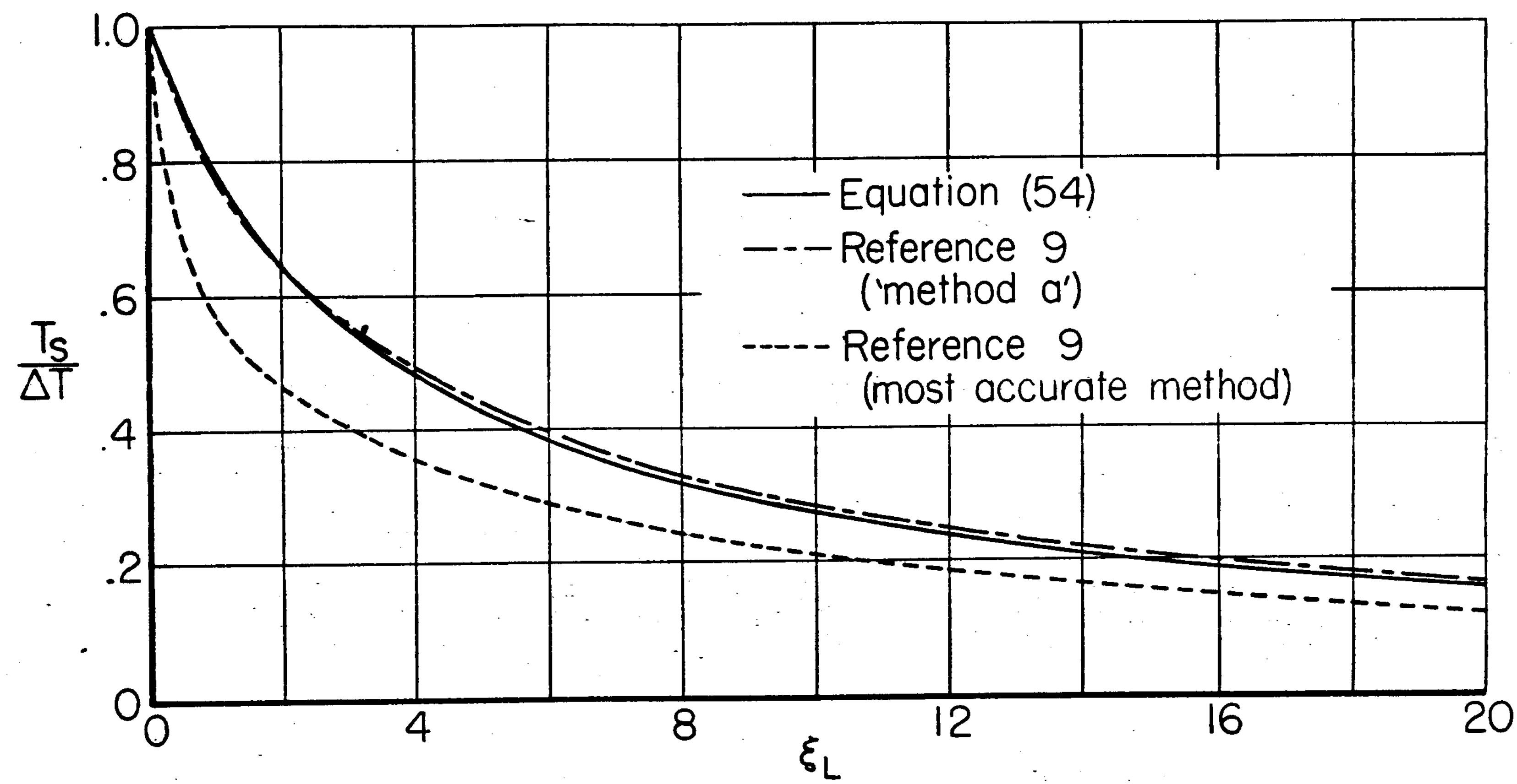
1.0
.8
.6
.4
.2
0
$\frac{T_s}{\Delta T}$
0
4
8
12
16
20
ξ_L
Equation (54)
Reference 9
('method a')
Reference 9
(most accurate method)

Fig. 4